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Coherent optical implementations of the fast Fourier transform and their comparison to the optical implementation of the quantum Fourier transform

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Introduction

- Coherent optical implementation of the Discrete Fourier transform (DFT)
- Coherent matrix-vector multiplier
- DFT as a Unitary Operation
- Fast Fourier transform (FFT) decomposition of DFT
- Coherent optical implementation based on FFT signal flow diagram

Introduction

- Electro-optical implementation of the FFT 'Butterfly' operation
- The quantum Fourier transform (QFT)
- Similarities and differences of coherent optical FFT to the QFT
- Use of the QFT in Shor's algorithm for large number factorisation
- Conclusions

Coherent optical implementation of the Discrete Fourier transform (DFT)

Forward DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$$

$$k = 0,1,2,\ldots,N-1$$

Inverse DFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}}$$

$$n = 0,1,2,\ldots,N-1$$

Write:

$$e^{-j\frac{2\pi nk}{N}} = W_N^{nk}$$

N^2 complex multiplications for direct evaluation

Coherent optical implementation of the Discrete Fourier transform (DFT)

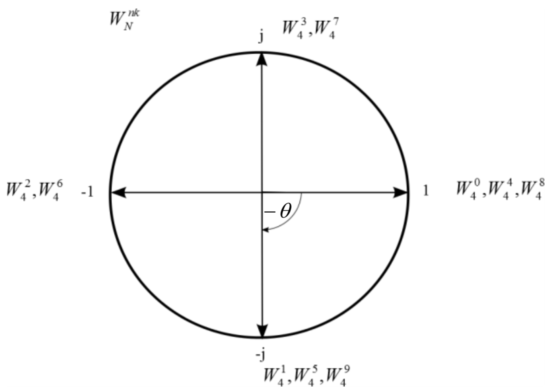
Can thus be written in matrix-vector form (for *N*=4):

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

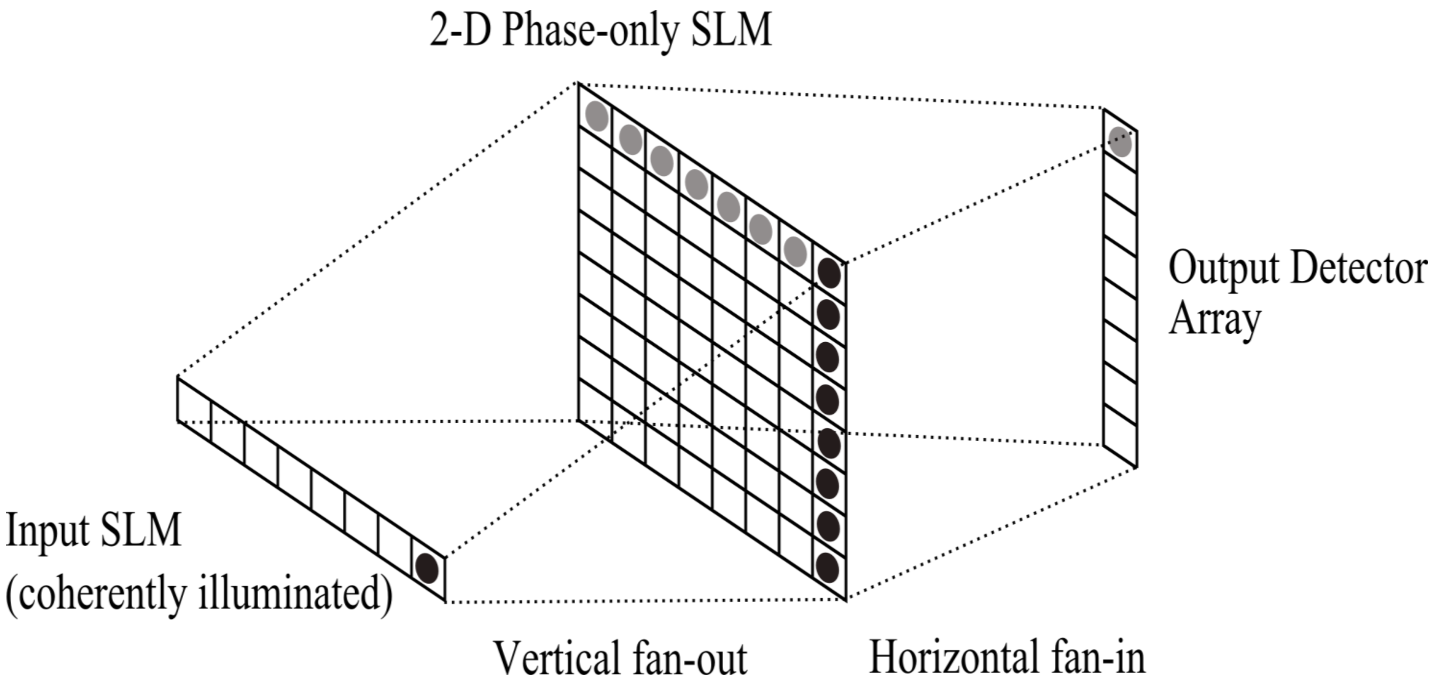
where the matrix **W(*k*,*n*)** can be expressed in terms of phase only retardations:

$$W(k,n) = \begin{bmatrix} e^{j0} & e^{j0} & e^{j0} & e^{j0} \\ e^{j0} & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{3\pi}{2}} \\ e^{j0} & e^{-j\pi} & e^{-j0} & e^{-j\pi} \\ e^{j0} & e^{-j\frac{3\pi}{2}} & e^{-j\pi} & e^{-j\frac{\pi}{2}} \end{bmatrix}$$

Unit circle in the *z*-plane



Coherent matrix-vector multiplier



Coherent matrix-vector multiplier for the calculation of the DFT

DFT as a Unitary Operation

A unitary operation transforms one (complex) vector to another by multiplication with a matrix that has the property:

$$\mathbf{W}\mathbf{W}^{\Diamond} = \mathbf{I}$$

where the $\Diamond$ superscript indicates the conjugate transpose of the matrix.

Thus computation of the DFT can be implemented as a reversible, non-dissipative operation.

Cooley-Tukey FFT decomposition of DFT (decimation in time); *N* a power of 2.

Half length DFTs of even and odd sequences:

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_{N/2}^{kn}+W_N^k\sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_{N/2}^{kn}$$

which can be written for $k=0,1,2,\ldots,\dfrac{N}{2}-1$:

$$X(k)=X_1(k)+W_N^k X_2(k)$$

by using the relation in W_N:

$$W_N^2=\left(e^{-j\frac{2\pi}{N}}\right)^2=e^{-j\frac{2\pi}{N/2}}=W_{N/2}$$

FFT decomposition of DFT

Using the symmetry in W_N: $W_N^{k-\frac{N}{2}} = -W_N^k$

we also have:

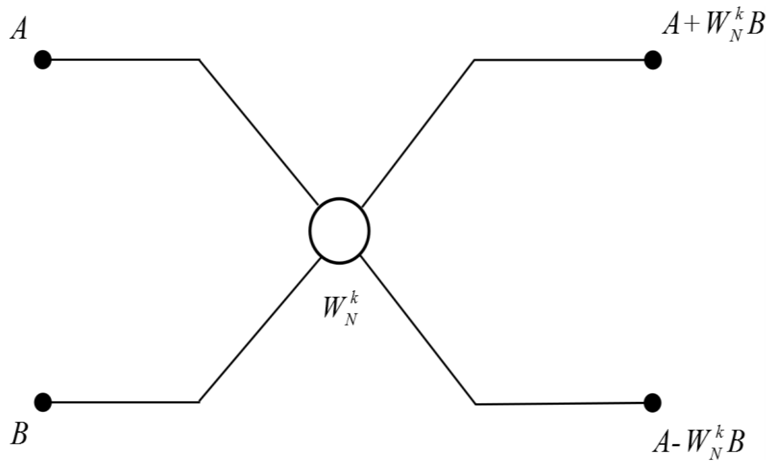
$$X(k) = X_1(k - N/2) - W_N^k X_2(k - N/2)$$

for values where: $\frac{N}{2} < k \leq N - 1$

Requires $N^2/2 + N/2$ complex multiplications

FFT decomposition of DFT

Splitting into an even and odd sequence is repeated until there are *N*/2 2-point DFTs which can each be represented by the FFT Butterfly signal flow diagram:

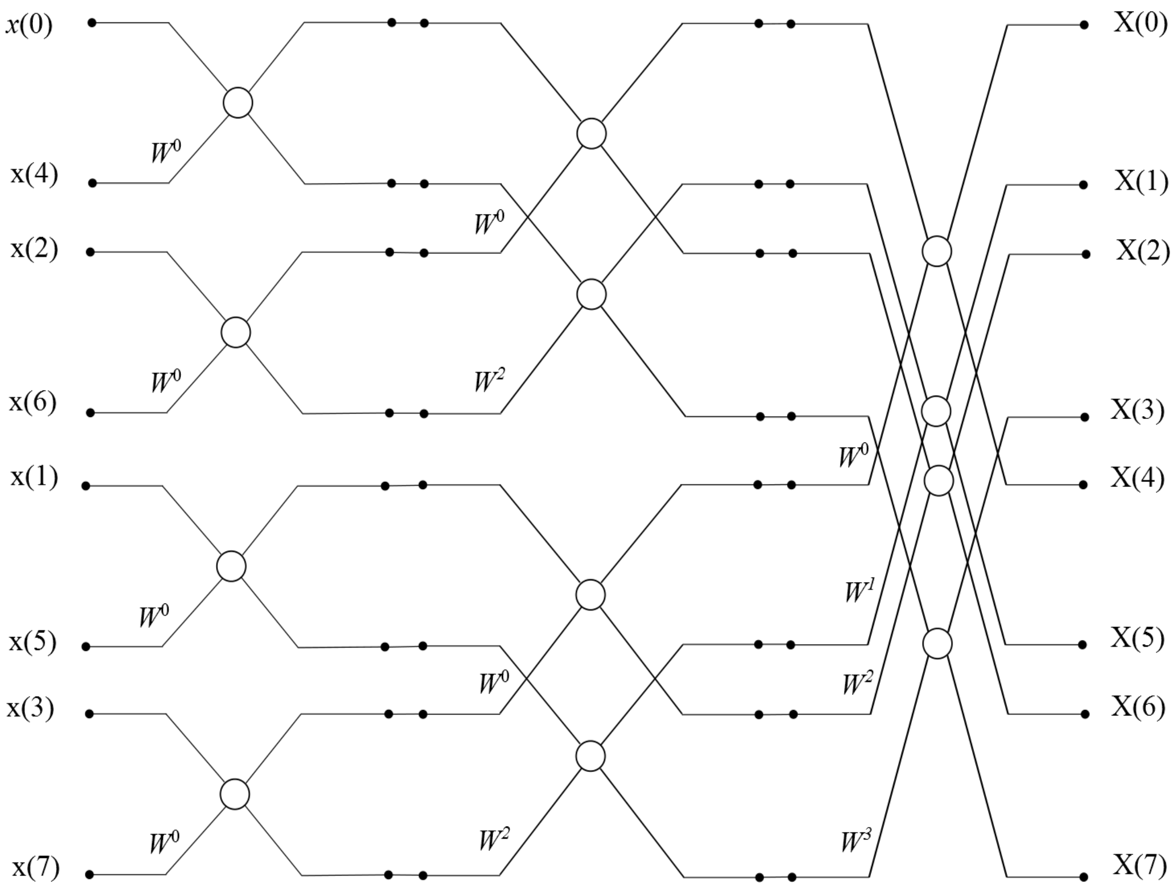


where for the 2-point transform W_N^k becomes W_N^0 i.e. unity.

Note that the computation involved in the Butterfly operation is Unitary.

FFT decomposition of DFT (decimation in time)

Larger FFTs can then be built up from the basic Butterfly operations e.g. for an 8-point decimation in time FFT the signal flow graph is as shown below.



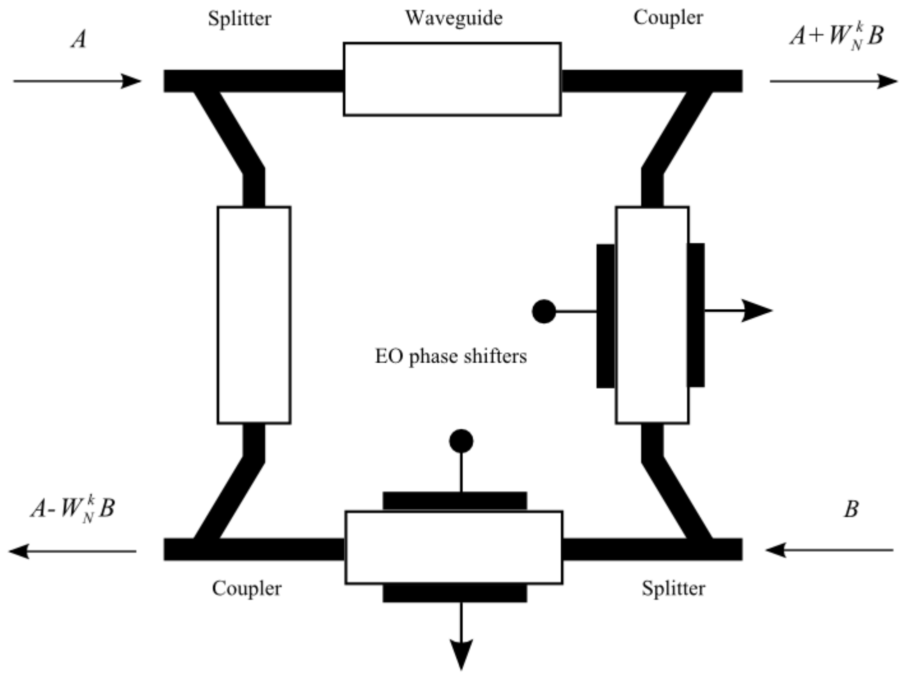
Coherent optical implementation based on FFT signal flow diagram

Siegman (*Optics Letters*, 2001) suggested fibre optic implementation of FFT based DFT employing 50:50 FO couplers to implement the Butterfly operations

Another possibility is to employ slab waveguides integrated to a ‘hybrid device’ as used in fibre optics coherent detection systems

Electro-optical implementation of the FFT Butterfly operation

Matrix-vector operation describing the hybrid: $$\begin{bmatrix} E_{o1} \\ E_{o2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{-j\phi} \end{bmatrix} \begin{bmatrix} E_{i1} \\ E_{i2} \end{bmatrix}$$



Hybrid device for coherent addition with controlled relative phase delays

The quantum Fourier transform (QFT)

The QFT acts on a wavefunction of, for example, four entangled 'qubits' (i.e. a one and zero superposition state at each bit location) described by the wavefunction or state vector:

$$\left|\psi\right\rangle = \sum_{n=0}^{N-1} x_n \left|n\right\rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The wavefunction is the superposition of four qubits each weighted by a probability x_n representing the value of the input signal at that 'qudit' location:

$$\left|\psi\right\rangle = x_{00}\left|00\right\rangle + x_{01}\left|01\right\rangle + x_{10}\left|10\right\rangle + x_{11}\left|11\right\rangle$$

The quantum Fourier transform (QFT)

The QFT of the state vector may then be computed:

$$\left|\Psi\right\rangle = \sum_{n=0}^{N-1} e^{-j\frac{2\pi nk}{N}} \left|\psi\right\rangle$$

which, since this is a unitary operation, maintains the wavefront superposition state but transforms it to the Fourier coefficients corresponding to the input wavefunction.

Thus we now have:

$$\left|\Psi\right\rangle = X_{00}\left|00\right\rangle + X_{01}\left|01\right\rangle + X_{10}\left|10\right\rangle + X_{11}\left|11\right\rangle$$

where the X_n are the complex coefficients corresponding to the complex Fourier components at that qudit location in the output array.

However, since they comprise the overall wavefunction, they will not be directly accessable to measurement. Rather the probability of detection, by a single photodetection event, will be given by $|\Psi|^2$.

Similarities and differences of coherent optical FFT to the QFT

- If the input wavefunction is periodic, $|\Psi|^2$ will have a peak in its probability distribution at the output location corresponding to this.

- Thus repeated application of the QFT will yield more detection events at this location and hence allow determination of the periodicity, *r*.

- Thus the QFT is more powerful than the FFT in that it can process 2^N (binary) inputs in parallel with effectively the same complexity of hardware structure (and so is exponentially faster in computation).

- However, the FFT yields *N* complex frequency components at its output whereas the QFT produces a probability distribution only which collapses to a single photon detection event upon measurement.

QFT implementation

An FFT-like decomposition of the QFT can be made using the Hadamard gate as the basic operation:

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

This arrangement will act on a single qubit state to give:

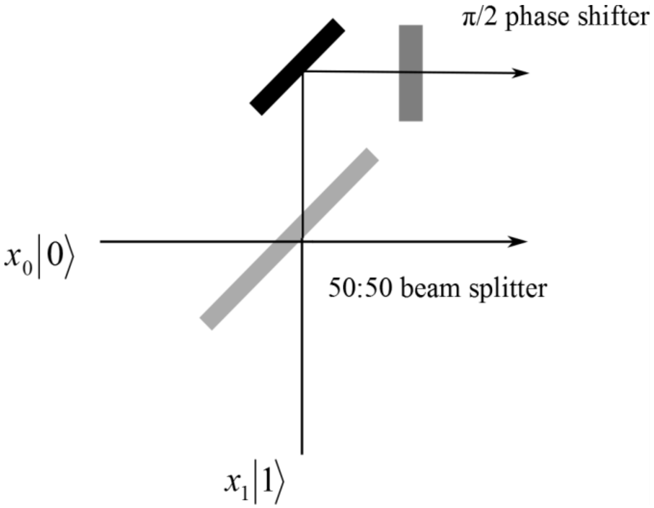
$$\mathbf{H}\left[x_0|0\rangle + x_1|1\rangle\right] = \frac{1}{\sqrt{2}}\left(x_0 + x_1\right)|0\rangle + \frac{1}{\sqrt{2}}\left(x_0 - x_1\right)|1\rangle$$

Thus it can be seen that the Hadamard transform performs a 2-point QFT by implementing the basic FFT building block of the Butterfly operation i.e. the subtraction and addition of the two input signals, albeit in a superposition state.

QFT optical implementation

The basic operation comprising the Hadamard transform is optically implementable with a 50:50 beam splitter together with an additional $\pi/2$ phase shift in one of the beams as shown below (Barak and Ben-Aryeh, *JOSA* B, 2007).

This arrangement is also used in fibre optic communication coherent detection systems to realise a 180^0 hybrid.



Optical implementation of Hadamard gate

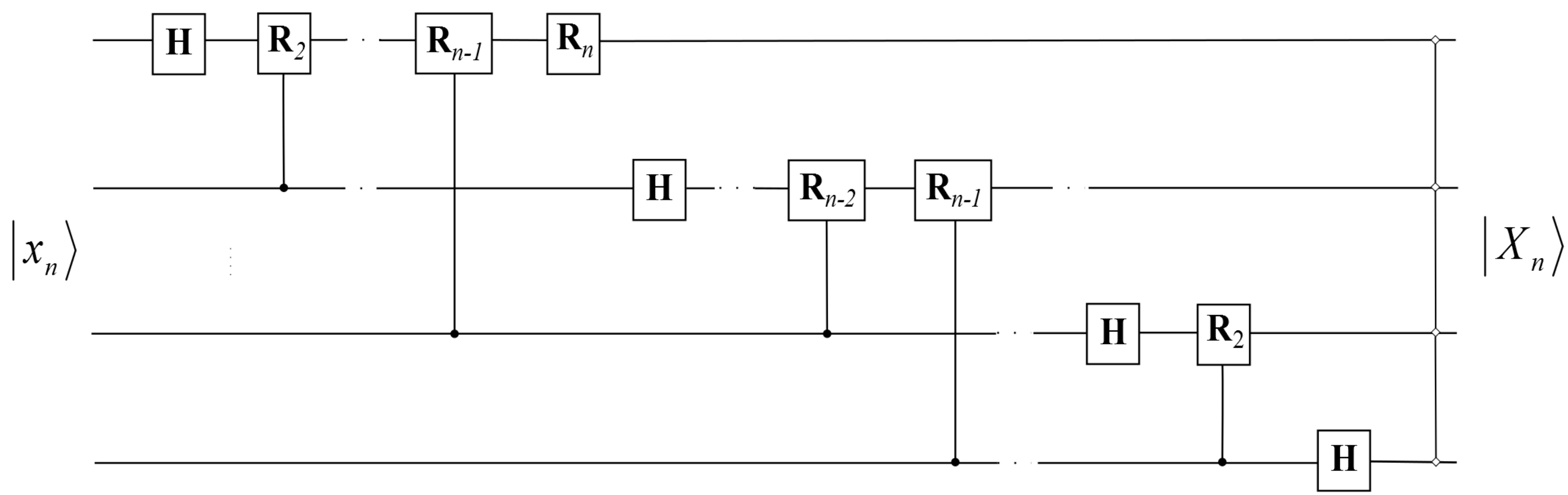
QFT implementation

Larger count QTFs can be implemented by the same basic operation with the additional inclusion of controlled phase rotation gates described by matrices of the form:

$$R_n = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & e^{-j\frac{2\pi}{2^n}} \end{bmatrix}$$

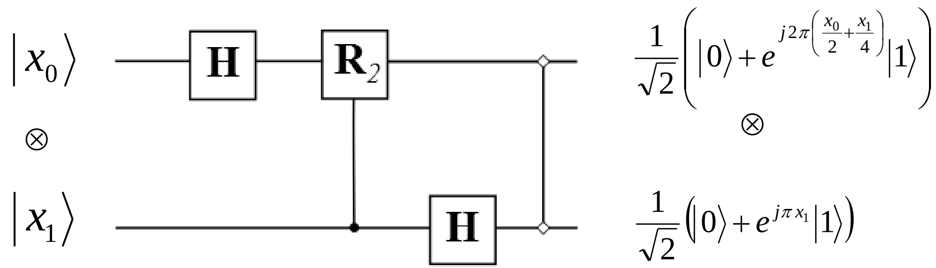
QFT implementation

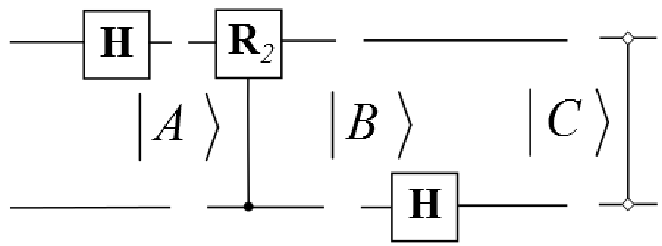
This allows the QFT structure for a *n*-qubit input to be represented compactly as shown below:



Two Qubit QFT

The circuit to implement the two-qubit QFT is:





with $|A\rangle = U_1|x\rangle, |B\rangle = U_2|A\rangle, |C\rangle = U_3|B\rangle$ and $|X\rangle = U_4|C\rangle$

to, overall, yield: $F = U_4U_3U_2U_1$

Two Qubit QFT

The unitary matrices for *n* = 2 are:

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad R_2 = U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-j\frac{\pi}{2}} \end{pmatrix} \quad U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to, overall, yield: $$F = U_4 U_3 U_2 U_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

Conclusions

- Coherent optical implementation of the DFT and FFT can be accomplished with pure phase retardations
- There is no dissipation involved in the computation and so it is reversible
- Thus, formally, the transformation matrix is a unitary matrix and so the DFT/FFT is a unitary operation.
- Thus a wavefunction can undergo Fourier transformation without collapse.

Conclusions

- The coherent optical and quantum optical DFT/FFT can be implemented with linear optical hardware structures.

- Use a converging lens?

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